Last class:

$f : I \to \mathbb{R}$ 1-1 differentiable, I open interval

$\Rightarrow$ can define inverse function: $f^{-1} : J = f(I) \to I$
as follows:

If $y \in J \Rightarrow y = f(x)$ for a unique $x \in I$

$\Rightarrow$ $\boxed{f^{-1}(y) = x \iff y = f(x)}$

shown in 142A: f^{-1} is continuous

Theorem: If f differentiable at x_0 with $f'(x_0) \neq 0$

$\Rightarrow$ f^{-1} " " $y_0 = f(x_0)$

and $(f^{-1})'(y_0) = \dfrac{1}{f(x_0)}$

Proof. need to show:

If (y_n) sequence with $y_n \to y_0$

$$\Rightarrow \lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}$$

in particular, limit exists!

let $\boxed{x_n = f^{-1}(y_n) \Leftrightarrow f(x_n) = y_n}$

f^{-1} continuous $\Rightarrow \lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y_0) = x_0$

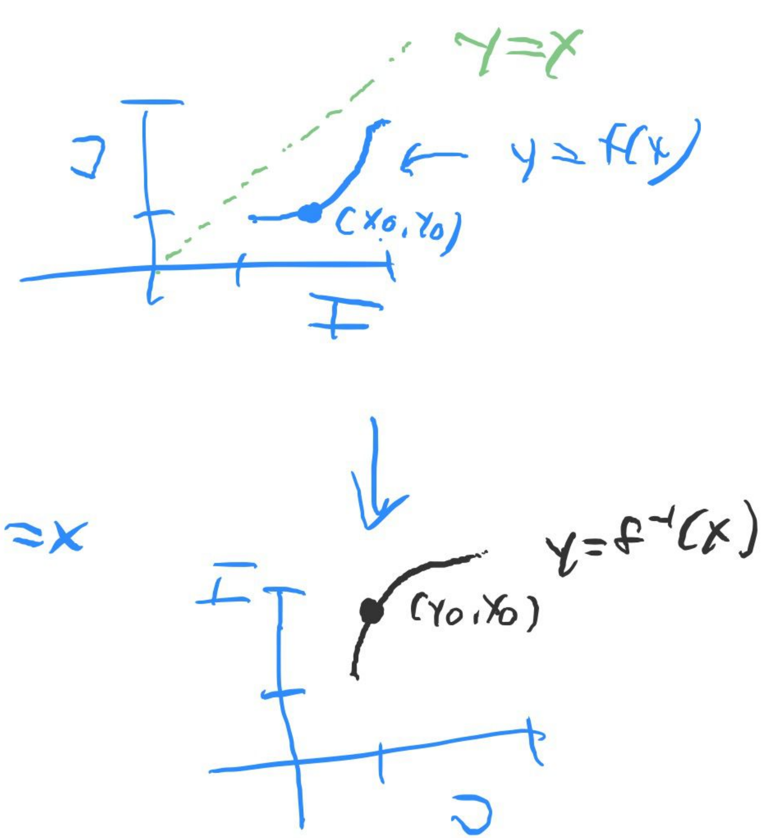
$$\Rightarrow \lim_{n\to\infty} x_n = x_0$$

$$\Rightarrow \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{x_n - x_0}{f(x_n) - f(x_0)}$$

$$\Rightarrow \lim_{n\to\infty} \text{"} = \lim_{n\to\infty} \text{"} = \frac{1}{f'(x_0)}$$ ✓

Geometric Interpretation:

If we can draw the graph of f



$\Rightarrow$ the graph of f^{-1} is obtained by reflecting graph of f at axis $y=x$

$\Rightarrow$ slope at (x_0,y_0) for f

$\rightarrow$ reciprocal of slope for f^{-1}

Hospital's Rule

need a generalization of MVT:

Theorem f, g cont. on $[a,b]$, differentiable on (a,b)

$\Rightarrow \exists\, x_0 \in (a,b)$ s.t.

$$f'(x_0)\,(g(b)-g(a)) = g'(x_0)\,(f(b)-f(a))$$

proof. use Rolle's theorem for

$$\boxed{h(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a))}$$

check! $h(a) \overset{!}{=} f(a)g(b) - g(a)f(b) \overset{!}{=} h(b)$

easy calculation

h differentiable $\Rightarrow \exists\, x_0$ s.t.

$h'(x_0) = 0$ by Rolle's theorem. But $h'(x_0) = f'(x_0)(g(b)-g(a)) - g'(x_0)(f(b)-f(a)) = 0 \Rightarrow$ claim

Remark: If $g(b)-g(a) \neq 0 \Rightarrow \exists x_0 \in (a,b)$

$$\Rightarrow \boxed{\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

De L'Hospital's Theorem.

know: If $\lim\limits_{x \to s} f(x) = f(s)$, $\lim\limits_{x \to s} g(x) = g(s)$

and $g(s) \neq 0 \Rightarrow \lim\limits_{x \to s} \frac{f(x)}{g(x)} = \frac{f(s)}{g(s)}$

Question: what if $\lim\limits_{x \to s} f(x) = 0 = \lim\limits_{x \to s} g(x)$?

similarly: what if $\lim\limits_{x \to s} f(x) = \infty = \lim\limits_{x \to s} g(x)$?

$$\Rightarrow \lim_{x \to s} \frac{f(x)}{g(x)} = \lim_{x_0 \to s} \frac{f'(x_0)}{g'(x_0)} = L$$

only thing to be checked: $g(x) - g(s) \neq 0$!

true for x near enough because $\boxed{g(x) - g(s) = g'(\tilde{x})\,(x-s)}$

as $\lim_{\tilde{x} \to s} \frac{f'(\tilde{x})}{g'(\tilde{x})}$ exists $\Rightarrow$ $g'(\tilde{x}) \neq 0$ near s

$\to$ $g(x) - g(s) \neq 0$

case 2 $\quad s = -\infty$

use same idea except that we can not use $f(s) = f(-\infty)$, $g(s) = g(-\infty)$

instead: approximate it via $f(y)$, $g(y)$ for y sufficiently large and negative

Theorem (L'Hospital's Rule)

Assume

- f, g differentiable near s — s could also be $\pm\infty$ (enough if only on one side of s e.g. for $x > s$, x near s)
- $\lim_{x \to s} f(x) = 0 = \lim_{x \to s} g(x)$

If $\lim_{x \to s} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to s} \frac{f(x)}{g(x)} = L$

Proof <u>case 1</u> s a real number $\qquad f(s) = 0 = g(s)$

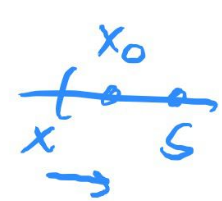
$$\xrightarrow{\lim_{x \to s}} \quad \frac{f(x)}{g(x)} = \frac{f(x) - f(s)}{g(x) - g(s)} = \frac{f'(x_0)}{g'(x_0)}$$

if $g(x) - g(s) \neq 0$ by gen. MVT

for some x_0 between x and s

(assume $x < s$ for simplicity)

obviously: $\boxed{\text{if } x \to s \implies x_0 \to s}$



if $L = \lim_{x \to -\infty} \frac{f'(x)}{g'(x)}$

$$0 \leq \lim_{x \to -\infty} \left| \frac{f(x)}{g(x)} - L \right| \leq \lim_{x \to -\infty} \left| \frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)} + \frac{f(x)-f(y)}{g(x)-g(y)} - L \right|$$

$$\leq \lim_{x \to -\infty} \underbrace{\left| \frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)} \right|}_{< \varepsilon/2} + \underbrace{\left| \frac{f(x)-f(y)}{g(x)-g(y)} - L \right|}_{< \varepsilon/2}$$

for y big enough $f(y), g(y)$ close to 0

using MVT as in Case 1

$$\Rightarrow \lim \left| \frac{f(x)}{g(x)} - L \right| = 0$$

$\Rightarrow$ claim.

Examples

① $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$

② can apply theorem repeatedly!

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{+\sin x}{2x} \underset{①}{=} \frac{1}{2}$$

③ need to check assumptions!

e.g. $\lim_{\substack{x \to 0 \\ x>0}} \frac{\ln x}{x}$ $\boxed{\text{"}=\text{"} \lim_{\substack{x \to 0 \\ x>0}} \frac{1/x}{1} = \infty}$ wrong

But for small $x>0$ $\ln x < 0 \Rightarrow \frac{\ln x}{x} < 0$

$\Rightarrow$ Limit should be negative!

indeed $\lim_{x \to 0} \left|\frac{\ln x}{x}\right| = \lim_{x \to 0} \underbrace{|\ln x|}_{\Downarrow \atop -\infty} \underbrace{\left|\frac{1}{x}\right|}_{\Downarrow \atop \infty} = \infty$

i.e. $\boxed{\lim \frac{\ln x}{x} = -\infty}$

if $x \to 0$ $\qquad \ln x \longrightarrow -\infty$

$\qquad\qquad\qquad x \longrightarrow 0$

assumptions for L'Hospital's theorem NOT given